Double Integrals over General Regions Math 261

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Double Integrals over Rectangles

Notation: we denote rectangles as the *cartesian* product of two intervals.

$$R = [a, b] \times [c, d] = \{(x, y) : a \leq x \leq b \text{ and } c \leq y \leq d\}$$

1. Let $R = [0, 3] \times [0, 2]$. Compute the following integral

$$\iint_R xy^3 \, dA$$

$$= \int_0^3 \int_0^2 x y^3 \, dy \, dx = \int_0^3 \left[x \frac{y^4}{4} \right]_0^2 dx$$

$$= \int_0^3 \left[x \, \frac{2^4}{2^2} - x \cdot \frac{0^4}{2^2} \right] dx$$

$$= \int_0^3 4x \, dx = \frac{\cancel{4}x^2}{\cancel{2}} \Big]_0^3 = 2 \cdot 3^2 - 2 \cdot 0^2$$

$$= 18$$

2. Let $R = [1, 2] \times [0, \frac{\pi}{3}]$. Compute the following integral

$$\iint_R x \cos(2y) \, dA$$

$$= \int_1^2 \left[\int_0^{\pi/3} x \cdot \cos(2y) \, dy \right] dx = \int_1^2 \left[x \cdot \frac{\sin(2y)}{2} \right]_0^{\pi/3} dx$$

$$= \int_1^2 \left[\frac{x \cdot \underbrace{\sin(\frac{2\pi}{3})}_{=\frac{\sqrt{3}}{2}}}{2} - \underbrace{\frac{x \cdot \sin(0)}{2}}_{=0} \right] dx$$

$$= \left[\frac{x^2}{2} \cdot \frac{\sin(\frac{2\pi}{3})}{2} \right]_1^2 = \frac{2^2}{2} \cdot \frac{\frac{\sqrt{3}}{2}}{2} - \frac{1^2}{2} \cdot \frac{\frac{\sqrt{3}}{2}}{2}$$

$$\boxed{= \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}}$$

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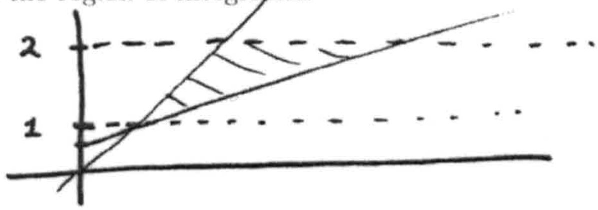
$x = 2y - 1 \Rightarrow y = \frac{1}{2}x + \frac{1}{2}$

$x = y \Rightarrow y = x$

2. Let $\displaystyle\int_1^2 \left[\int_y^{2y-1} 2x - y^2 \, dx\right] dy$

(a) Sketch the region of integration



(b) Compute the specified volume.

$$\int_1^2 \left[\int_y^{2y-1} 2\underline{\underline{x}} - y^2 \, d\underline{\underline{x}}\right] dy$$

$$= \int_1^2 \left[\frac{\cancel{2}x^2}{\cancel{2}} - y^2 x\right]_y^{2y-1} dy$$

$$= \int_1^2 \left((2y-1)^2 - y^2(2y-1)\right) - \left(y^2 - y^2 \cdot y\right) \, dy$$

$$= \int_1^2 4y^2 - 4y + 1 - \underline{\underline{2y^3}} + \cancel{y^2} - \cancel{y^2} + \cancel{y^3} \, dy$$

$$= \int_1^2 4y^2 - 4y + 1 - y^3 \, dy$$

$$= \left.\frac{4y^3}{3} - \frac{4y^2}{2} + y - \frac{y^4}{4}\right]_1^2$$

$$= \left(\frac{4 \cdot 2^3}{3} - \frac{4 \cdot 2^2}{2} + 2 + \frac{2^4}{4}\right) - \left(\frac{4 \cdot 1}{3} - \frac{4 \cdot 1}{2} + 1 - \frac{1}{4}\right)$$ good enough for test

$= \ldots$

[redacted]

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4. Let $f(x,y) = 2y - x^2$. Compute the net volume under f on the region D bounded between $y = 3x$ and $y = x^2$.

intersect when

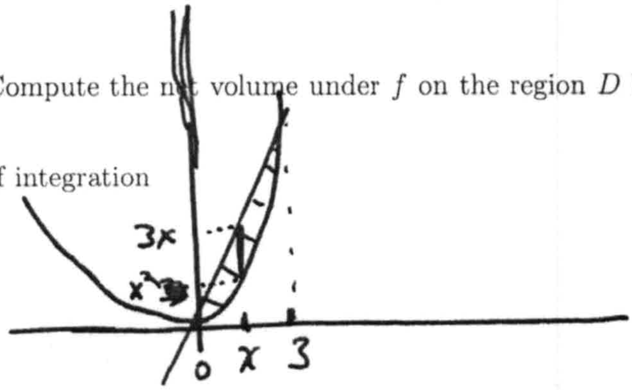
$3x = x^2$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0$ or $x = 3$

   (a) Sketch the region of integration



   (b) Write the volume under f on D as a Type I integral (where functions give the boundary for the top and bottom of D).

• inside will be "area of slice at x"

will be $\int_{y=x^2}^{y=3x} f(x,y)\,dy$

---

net volume under f on D $= \int_{x=0}^{x=3} \int_{y=x^2}^{y=3x} 2y - x^2 \, dy \, dx$

   (c) Compute the volume under $f(x,y)$ on the specified region

$$= \int_0^3 \left[\frac{\cancel{2}y^2}{\cancel{2}} - x^2 y \right]_{y=x^2}^{y=3x} dx$$

$$= \int_0^3 \left((3x)^2 - x^2\cdot(3x)\right) - \left((x^2)^2 - x^2\cdot x^2\right) dx$$

$$= \int_0^3 9x^2 - 3x^3 \cancel{- x^4 + x^4} \, dx$$

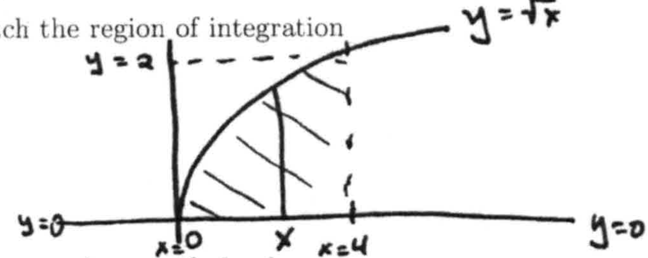
$$= \frac{9x^3}{3} - 3\,\frac{x^4}{4}\Big]_0^3 = \left(\frac{\cancel{9}\cdot 3^3}{\cancel{3}} - 3\cdot\frac{3^4}{4}\right) - (0 - 0)$$

$$= 3^4 - \frac{1}{4}\cdot 3^5$$

Name: ______ area of slice at x ______ Section: ______________

5. Let $\displaystyle\int_{x=0}^{x=4}\int_{y=0}^{y=\sqrt{x}} xy\; dy\; dx$

(a) Sketch the region of integration



(b) Compute the specified volume.

$$\int_0^4 \left[\int_0^{\sqrt{x}} xy\, dy\right] dx$$

$$= \int_0^4 \left[x\frac{y^2}{2}\right]_{y=0}^{\sqrt{x}} dx$$

$$= \int_0^4 \frac{x\overbrace{(\sqrt{x})^2}^{x}}{2} - \frac{x\cdot 0^2}{2}\, dx$$

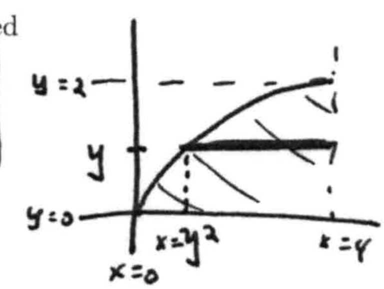
$$= \int_0^4 \frac{x^2}{2}\, dx$$

$$= \left.\frac{x^3}{6}\right]_0^4 = \frac{4^3}{6} - \frac{0^3}{6} = \frac{4^3}{6} = \frac{2^6}{2\cdot 3} = \frac{2^5}{3}$$

(c) Write an *equivalent* integral with the order of integration reversed

want inside to give area of slice at y

area under slice at y $= \displaystyle\int_{x=y^2}^{x=4} f(x,y)\, dx$



Volume under f

$$= \int_{y=0}^{y=2}\int_{x=y^2}^{x=4} xy\; dx\; dy$$

$y=\sqrt{x} \Rightarrow x = y^2$

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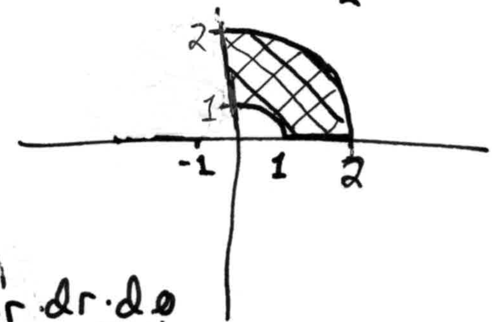
Integration in Polar Coordinates

1. Sketch the region defined by the polar rectangle $\mathcal{R}$ with $r \in [1, 2]$ and $\theta \in [0, \pi/2]$. Then compute the net volume under $f(x, y) = 4xy + 3x$ on $\mathcal{R}$.



$$\iint_{\mathcal{R}} f(x,y)\,dA$$

$$= \int_{\theta=0}^{\theta=\pi/2} \int_{r=1}^{r=2} \Big(4(r\cos\theta)(r\sin\theta) + 3r\cdot\cos\theta\Big)\underbrace{r\cdot dr\cdot d\theta}_{dA}$$

$$\begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \\ dA = r\cdot dr\cdot d\theta \end{cases}$$

$$= \int_0^{\pi/2}\left[\int_1^2 4r^3\cos\theta\sin\theta + 3r^2\cos\theta\, dr\right]d\theta$$

$$= \int_0^{\pi/2}\left[\cancel{4}\cdot\frac{r^4}{\cancel{4}}\cos\theta\sin\theta + \cancel{3}\cdot\frac{r^3}{\cancel{3}}\cos\theta\right]_1^2 d\theta$$

$$= \int_0^{\pi/2}(2^4\cos\theta\sin\theta + 2^3\cos\theta) - (1^4\cos\theta\sin\theta + 1^3\cos\theta)\, d\theta$$

$$= \int_0^{\pi/2}(2^4-1)\cos\theta\,\overbrace{\sin\theta\, d\theta}^{du} + \int_0^{\pi/2}(2^3-1)\cos\theta\, d\theta$$

$u = \sin\theta$ $\qquad$ $\theta = 0 \Rightarrow u = \sin 0 = 0$

$\frac{du}{d\theta} = \cos\theta$ $\qquad$ $\theta = \pi/2 \Rightarrow u = \sin\frac{\pi}{2} = 1$

$du = \cos\theta\, d\theta$

$$= \int_{u=0}^{u=1}(2^4-1)\,u\,du + (2^3-1)\Big[\sin\theta\Big]_0^{\pi/2}$$

$$= \Big((2^4-1)(1) - 0\Big) + \Big((2^3-1)(1) - 0\Big)$$

$$= 2^4 + 2^3 - 2$$

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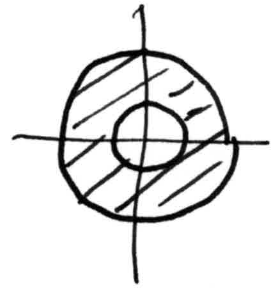
2. Suppose that a washer $\mathcal{D}$ with an inner radius of 1m and an outer radius of 2m is centered at $\mathcal{O}$. The density of $\mathcal{D}$ is given by $d(x, y) = x^2 + y^2$. Compute the mass of the washer.

$$\mathcal{D} \begin{cases} 1 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{cases}$$



$$\begin{cases} x = r \cdot \cos\theta \\ y = r \cdot \sin\theta \\ dA = r \cdot dr\, d\theta \end{cases} \quad \Big\} r^2 = x^2 + y^2$$

$$\iint_R x^2 + y^2 \, dA = \int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=2} r^2 \cdot \underbrace{r\, dr\, d\theta}_{dA}$$

$$= \int_0^{2\pi} \left[\int_1^2 r^3 \, dr \right] d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^4}{4} \right]_1^2 d\theta$$

$$= \int_0^{2\pi} \frac{2^4}{4} - \frac{1^4}{4} \, d\theta$$

$$= \left[\frac{2^4 - 1}{4} \cdot \theta \right]_0^{2\pi}$$

$$= \frac{2^4 - 1}{4} 2\pi - 0$$

$$= \frac{2^4 - 1}{2} \pi \text{ Kg}$$

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Triple Integrals

1. Compute $\iiint_{\mathcal{B}} \frac{2x+6y}{z}\,dV$ on the box $\mathcal{B} = [0,3] \times [1,2] \times [1,e]$.

$$= \int_{x=0}^{x=3} \int_{y=1}^{y=2} \left[\int_{z=1}^{z=e} (2x+6y) \cdot \frac{1}{z}\,dz \right] dy\,dx$$

$$= \int_0^3 \int_1^2 \Big[(2x+6y) \cdot \ln(z)\Big]_{z=1}^{z=e} dy\,dx$$

$$= \int_0^3 \int_1^2 (2x+6y) \cdot \underset{\overset{\shortparallel}{1}}{\ln(e)} - (2x+6y) \cdot \underset{\overset{\shortparallel}{0}}{\ln(1)}\; dy\,dx$$

$$= \int_0^3 \left[\int_1^2 (2x+6y)\,dy \right] dx$$

$$= \int_0^3 \left[2xy + \frac{\cancelto{3}{6}y^2}{\cancel{2}} \right]_{y=1}^{y=2} dx$$

$$= \int_0^3 \Big(\underbrace{2x \cdot 2}_{4x} + \underbrace{3 \cdot 2^2}_{12}\Big) - \Big(\underbrace{2 \cdot x \cdot 1}_{2x} + \underbrace{3 \cdot 1^2}_{3}\Big)\; dx$$

$$= \int_0^3 2x + 9\,dx$$

$$= \left[\frac{\cancel{2}x^2}{\cancel{2}} + 9x \right]_0^3 = (3^2 + 9 \cdot 3) - (0^2 + 9 \cdot 0)$$

$$= 9 + 27$$

$$= 36$$

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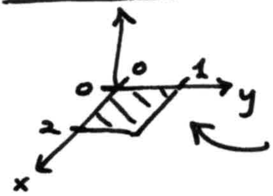
2. Let $f(x, y, z) = x$ be a density function. Compute the mass of the region bounded by the planes $z = x - 2y$ and $z = 2x + y$ above the rectangle $[0, 2] \times [0, 1]$.



"shadow" of desired region

$$\iiint f(x,y,z)\, dV$$

$$= \int_{x=0}^{x=2} \int_{y=0}^{y=1} \left[\int_{z=x-2y}^{z=2x+y} [x]\, dz \right] dy\, dx$$

$$= \int_0^2 \int_0^1 \left[xz \right]_{z=x-2y}^{z=x+y} dy\, dx$$

$$= \int_0^2 \int_0^1 x(x+y) - x(x-2y)\, dy\, dx$$

$$= \int_0^2 \int_0^1 \cancel{x^2} + xy - \cancel{x^2} + 2xy\, dy\, dx$$

$$= \int_0^2 \left[\int_0^1 3xy\, dy \right] dx$$

$$= \int_0^2 \left[3x \frac{y^2}{2} \right]_{y=0}^{y=1} dx = \int_0^2 3x \cdot \frac{1^2}{2} - 3x \cdot \frac{0^2}{2}\, dx$$

$$= \int_0^2 \frac{3}{2} x\, dx$$

$$= \left[\frac{3}{2} \frac{x^2}{2} \right]_0^2 = \frac{3}{4} \cdot 2^2 - \frac{3}{4} \cdot 0^2 = 3$$

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3. Compute the integral

$$\int_0^2 \int_0^{1-x} \left[\int_x^{x+y} x \; dz\right] dy\, dx$$

$$= \int_{x=0}^{x=2} \int_{y=0}^{y=1-x} \Big[x \; z\Big]_{z=x}^{z=x+y} dy\, dx$$

$$= \int_0^2 \int_{y=0}^{y=1-x} x\,(x+y) - x\,(x) \; dy\, dx$$

$$= \int_0^2 \left[\int_{y=0}^{y=1-x} \cancel{x^2} + xy - \cancel{x^2} \; dy\right] dx$$

$$= \int_0^2 \left[x \frac{y^2}{2}\right]_{y=0}^{y=1-x} dx$$

$$= \int_0^2 \frac{x}{2} \cdot (1-x)^2 \; dx$$

$$\boxed{\begin{aligned} \frac{x}{2}(1-x)^2 &= \frac{x}{2}(1-2x+x^2) \\ &= \frac{x}{2} - x^2 + \frac{x^3}{2} \end{aligned}}$$

$$= \int_0^2 \frac{x}{2} - x^2 + \frac{x^3}{2} \; dx$$

$$= \left[\frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8}\right]_0^2$$

$$= \left(\frac{2^2}{2^2} - \frac{2^3}{3} + \frac{2^4}{2^3}\right) - \left(0 - 0 + 0\right)$$

$$= 1 - \frac{8}{3} + 2 = 3 - \frac{8}{3}$$

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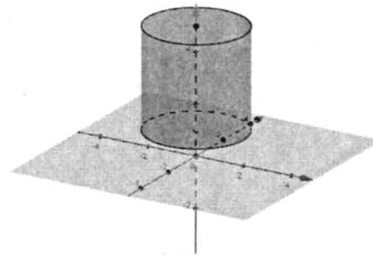
1. A certain density function is given by $f(x, y, z) = z\sqrt{x^2 + y^2}$ in kg/m^3.

Integrate f over the cylinder $\mathcal{W}$ with $x^2 + y^2 \leq 4$ and $1 \leq z \leq 5$.

in θ- coordinates

$$\mathcal{W} = \begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \\ 1 \leq z \leq 5 \end{cases}$$



Know:

$$\begin{cases} r = \sqrt{x^2+y^2} \\ x = r \cdot \cos\theta \\ y = r \cdot \sin\theta \\ z = z \\ dV = r \cdot dz\,dr\,d\theta \end{cases}$$

NOTICE: $f(x,y,z) = z \cdot \sqrt{x^2+y^2} = z \cdot r$

$$\iiint_W f(x,y,z)\,dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=1}^{5} \underbrace{(z \cdot r)}_{f} \cdot \underbrace{r \cdot dz\,dr\,d\theta}_{dV}$$

$$= \int_0^{2\pi} \int_0^2 \left[\int_1^5 z \cdot r^2\,dz \right] dr\,d\theta = \int_0^{2\pi} \int_0^2 \left[\frac{z^2}{2} r^2 \right]_{z=1}^{z=5} dr\,d\theta$$

$$= \int_0^{2\pi} \left[\int_0^2 \left(\frac{5^2 - 1^2}{2} \right) r^2\,dr \right] d\theta = \int_0^{2\pi} \left[\frac{5^2-1^2}{2} \cdot \frac{r^3}{3} \right]_{r=0}^{2} d\theta$$

$$= \int_0^{2\pi} \left(\frac{5^2-1^2}{2} \right) \cdot \frac{2^3}{3}\,d\theta = \left[\left(\frac{5^2-1^2}{2} \right) \cdot \frac{2^3}{3} \cdot \theta \right]_0^{2\pi} = 2\pi \cdot \frac{2^3}{3} \cdot \left(\frac{5^2-1^2}{2} \right)$$

$$= 64 \text{ kg.}$$

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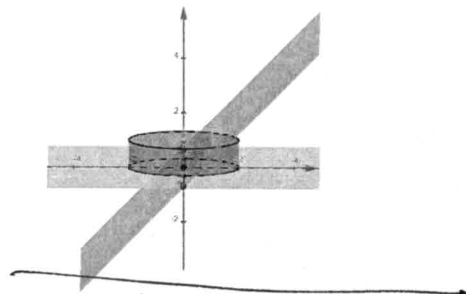
2. Set up and evaluate the integral of $f(x,y,z) = z$ on the cylinder $\mathcal{D}$ with $x^2 + y^2 \leq 4$ above the $x - y$ plane and below the plane $z = y$.

Recall
$$\begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \\ z = z \\ dV = r\cdot dz\cdot dr\cdot d\theta \end{cases}$$

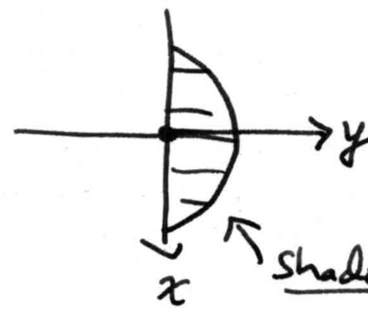


top view

NOTICE

$$D = \begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta \leq \pi \\ 0 \leq z \leq y = r\sin\theta \end{cases}$$

compute



Shadow of D.

$$\iiint_D f\,dV = \int_{\theta=0}^{\pi}\int_{r=0}^{2}\left[\int_{z=0}^{r\cdot\sin\theta} \underbrace{z}_{f}\cdot \underbrace{r\cdot dz}_{dV}\right] dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{2} \left[\frac{z^2 r}{2}\right]_{z=0}^{z=r\cdot\sin\theta} dr\,d\theta = \int_0^{\pi}\int_0^{2} \frac{(r\cdot\sin\theta)^2\cdot r}{2} - \frac{0}{2}\,dr\,d\theta$$

$$= \int_0^{\pi}\left[\int_0^{2} \frac{r^3\cdot\sin^2\theta}{2}\,dr\right]d\theta = \int_0^{\pi}\left[\frac{r^4}{4}\cdot\frac{\sin^2\theta}{2}\right]_{r=0}^{r=2} d\theta$$

$$= \int_0^{\pi} \frac{2^4}{2^2\cdot 2}\sin^2\theta\,d\theta \quad (=2)$$

$$\boxed{\text{Recall: } \sin^2\theta = \frac{1-\cos(2\theta)}{2}}$$

$$= \int_0^{\pi} \cancel{2}\,\frac{1-\cos(2\theta)}{\cancel{2}}\,d\theta = \left[\theta - \frac{\sin(2\theta)}{2}\right]_{\theta=0}^{\theta=\pi} = \left(\pi - \frac{\sin(2\pi)}{2}\right) - \left(0 - \frac{\sin(0)}{2}\right)$$

$$= \pi$$

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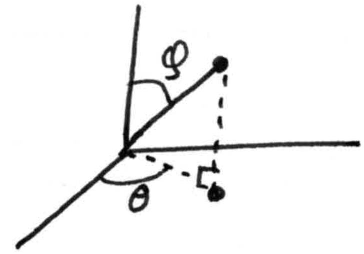
3. Use integration with spherical coordinates to compute the volume of a sphere of radius r.

$$\text{Sphere } S = \begin{cases} 0 \le \rho \le r \\ 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{cases}$$



Volume of S

$$= \iiint_S 1 \cdot dV = \int_{\rho=0}^{\rho=r} \int_{\varphi=0}^{\varphi=\pi} \left[\int_{\theta=0}^{\theta=2\pi} 1 \cdot \rho^2 \cdot \sin\varphi \cdot d\theta \right] d\varphi \cdot d\rho$$

$$= \int_0^r \int_0^\pi \Big[\rho^2 \cdot \sin\varphi \cdot \theta \Big]_{\theta=0}^{\theta=2\pi} d\varphi \, d\rho$$

$$= \int_0^r \left[\int_0^\pi \rho^2 \cdot \sin\varphi \cdot 2\pi \, d\varphi \right] \cdot d\rho$$

$$= \int_0^r \Big[\rho^2 \cdot (-\cos(\varphi)) \cdot 2\pi \Big]_{\varphi=0}^{\varphi=\pi} d\rho$$

$$= \int_0^r -\rho^2 \cos(\pi) \cdot 2\pi - (-\rho \cos(0)) \; d\rho$$

$$\boxed{\begin{array}{l} \cos(\pi) = (-1) \\ \cos(0) = +1 \end{array}}$$

$$= \int_0^r 2\rho \cdot 2\pi \; d\rho$$

$$= 2 \cdot \frac{\rho^3}{3} \cdot 2\pi \Big]_{\rho=0}^{\rho=r} = \frac{4\pi r^3}{3} - 0 = \boxed{\frac{4}{3}\pi r^3}$$

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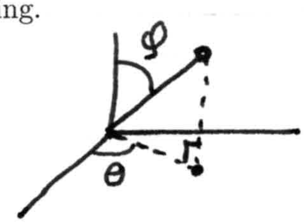
4. Suppose you buy a spherical bearing with radius $\rho = 2\ m$. The bearing's density is given by $f(x, y, z) = x^2 + y^2 + z^2$ in kg/m^3. Find the mass of the bearing.

$$S = \begin{cases} 0 \le \rho \le 2 \\ 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{cases}$$



Recall

$$\rho = \sqrt{x^2+y^2+z^2}$$

so

$$f(x,y,z) = \rho^2$$

$$\iiint_S f\,dV = \int_{\rho=0}^{\rho=2} \int_{\varphi=0}^{\varphi=\pi} \left[\int_{\theta=0}^{\theta=2\pi} \underbrace{\rho^2}_{f} \cdot \underbrace{\rho^2 \cdot \sin\varphi \cdot d\theta}_{dV} \right] d\varphi\, d\rho$$

$$= \int_0^2 \int_0^\pi \left[\rho^4 \cdot \sin\varphi \cdot \theta\right]_0^{2\pi} d\varphi\, d\rho = \int_0^2 \left[\int_0^\pi \rho^4 \cdot \sin\varphi \cdot 2\pi - 0\ d\varphi \right] d\rho$$

$$= \int_0^2 \left[\rho^4 (-\cos\varphi)\, 2\pi\right]_{\varphi=0}^{\pi} d\rho$$

$-\cos(\pi) = -(-1) = 1$

$-\cos(0) = -1$

$$= \int_0^2 \rho^4 \underbrace{(-\cos(\pi))}_{1}\, 2\pi - \rho \underbrace{(-\cos(0))}_{(-1)}\, 2\pi\ d\rho = \int_0^2 \rho^4 \cdot 2 \cdot 2\pi\ d\rho$$

$$= \left[\frac{\rho^5}{5} \cdot 4\pi\right]_{\rho=0}^{\rho=2} = \frac{2^5}{5} \cdot 4\pi \text{ kg.}$$